Remark on the Kato smoothing effect for Schrödinger equation with superquadratic potentials

Luc Robbiano and Claude Zuily

Résumé

The aim of this note is to extend recent results of Yajima-Zhang [Y-Z1, Y-Z2] on the $\frac{1}{2}$- smoothing effect for Schrödinger equation with potential growing at infinity faster than quadratically.

1 Introduction

The aim of this note is to extend a recent result by Yajima-Zhang [Y-Z1, Y-Z2]. In this paper these authors considered the Hamiltonian $H = -\Delta + V(x)$ where V is a real and C^∞ potential on $\mathbb{R}^n$ satisfying for some $m > 2$ and $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$,

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|},\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{N}^n, \tag{1.1}$$

$$\text{for large } |x|, \quad V(x) \geq C_1 |x|^m, \quad C_1 > 0, \tag{1.2}$$

and they proved the following. For any $T > 0$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ one can find $C > 0$ such that for all u_0 in $L^2(\mathbb{R}^n)$,

$$\int_0^T \|\chi (I - \Delta)^{\frac{1}{2m}} e^{-itH} u_0\|^2_{L^2(\mathbb{R}^n)} dt \leq C \|u_0\|^2_{L^2(\mathbb{R}^n)} \tag{1.3}$$

where Δ is the flat Laplacian. In this note, using the ideas contained in Doï [D3] we shall show that one can handle variable coefficients Laplacian with time dependent potentials, one can remove the condition (1.2), one can replace the cut-off function χ in (1.3) by $\langle x \rangle^{-\frac{1+\nu}{2}}$ with any $\nu > 0$ and finally that the weight $\langle x \rangle^{-\frac{1}{2}}$ is enough for the tangential derivatives.
When $V = 0$ the estimate (1.3) goes back to Constantin-Saut [C-S], Sjölin [S], Vega [V], Yajima [Y] who extended to the Schrödinger equation a phenomenon discovered by T. Kato [K] on the KdV equation. Later on their results where extended to the variable coefficients operators by Doï in a series of papers [D1, D2, D3, D4] which contained the case $m = 2$ of

Theorem 1.1 below.
Let us describe more precisely our result. It will be convenient to introduce the Hörmander's metric

$$g = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle \xi\rangle^2} \tag{1.4}$$

to which we associate the usual class of symbols $S(M,g)$ if M is a weight. Recall that $q \in S(M,g)$ iff $q \in C^\infty(\mathbb{R}^{2n})$ and

$$\forall \alpha,\beta \in \mathbb{N}^n\ \exists C_{\alpha\beta} > 0,\ |\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \leqq C_{\alpha\beta} M(x,\xi)\langle x\rangle^{-|\beta|}\langle \xi\rangle^{-|\alpha|},\ \forall (x,\xi) \in T^*(\mathbb{R}^n)$$

If $T > 0$ we shall set

$$S_T(M,g) = L^\infty([0,T], S(M,g)). \tag{1.5}$$

We shall consider here an operator P of the form

$$P = \sum_{j,k=1}^{n} (D_j - a_j(t,x)) g^{jk}(x)(D_k - a_k(t,x)) + V(t,x) \tag{1.6}$$

and we shall denote by p the principal symbol of P, namely

$$p(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k. \tag{1.7}$$

We shall make the following structure and geometrical assumptions.

Structure assumptions. We shall assume the following,

$$\begin{cases} (i) \text{ the coefficients } a_j, g^{jk},\ V \text{are real valued for } j,k = 1,...,n, \\ (ii)\ p \in S(\langle\xi\rangle^2, g) \text{ and } \nabla g^{jk}(x) = o(|x|^{-1}),\ |x| \to +\infty\ 1 \leqq j,k \leqq, n \\ (iii)\ a_j \in S_T(\langle x\rangle^{\frac{m}{2}}, g),\ 1 \leqq j \leqq n,\ V \in S_T(\langle x\rangle^m, g)\ m \geqq 2 \end{cases} \tag{1.8}$$

$$\exists \delta > 0,\ p(x,\xi) \geqq \delta|\xi|^2, \quad \forall (x,\xi) \in T^*(\mathbb{R}^n). \tag{1.9}$$

(1.10) For any fixed t in $[0,T]$ the operator P is essentially self adjoint on $L^2(\mathbb{R}^n)$

Geometrical assumptions. Let ϕ_t be the bicharacteristic flow of p. It is easy to see that under the conditions (1.8), (1.9) it is defined for all $t \in \mathbb{R}$. Let us set $S^*(\mathbb{R}^n) = \{(x,\xi) \in T^*(\mathbb{R}^n) : p(x,\xi) = 1\}$. Then we shall assume that,

$$\forall K \text{compact} \subseteqq S^*(\mathbb{R}^n)\ \exists t_K > 0 \text{ such that } \Phi_t(K) \cap K = \emptyset\ , \quad \forall t \geq t_K. \tag{1.11}$$

This is the so-called "non trapping condition" which is equivalent to the fact that if $\Phi_t(x;\xi) = (x(t),(\xi(t))$ then $\lim_{t\to+\infty} |x(t)| = +\infty$.

We shall consider $u \in C^1([0,T], \mathcal{S}(\mathbb{R}^n))$ and we set

$$f(t) = (D_t + P)u(t) \tag{1.12}$$

For $s \in \mathbb{R}$ let $e_s(x,\xi) = (1 + |\xi|^2 + |x|^m)^{\frac{s}{2}}$ and E_s be the Weyl quantized pseudo-differential operator with symbol e_s.
Our first result is the following.

Theorem 1.1 *Let $T > 0$. Let P be defined by (1.6) which satisfies (1.8), (1.9), (1.10),(1.11). Then for any $\nu > 0$ one can find $C = C(\nu, T) > 0$ such that for any $u \in C^1([0,T], \mathcal{S}(\mathbb{R}^n))$ and all t in $[0,T]$ we have,*

$$\|u(t)\|^2_{L^2} + \int_0^T \|\langle x\rangle^{-\frac{1+\nu}{2}} E_{\frac{1}{m}} u(t)\|^2_{L^2} dt \leqq \ C\ (\|u(0)\|^2_{L^2} + \int_0^T \|\langle x\rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|^2_{L^2} dt).$$

Here $L^2 = L^2(\mathbb{R}^n)$ and $f(t)$ is defined by (1.12).

Now even when P is the flat Laplacian it is known that the estimate in the above Theorem does not hold with $\nu = 0$. However we have the following result. Let us set

$$\ell_{jk} = \frac{x_j\xi_k - x_k\xi_j}{\langle x\rangle\langle\xi\rangle}, \quad 1 \leqq j,k \leqq n\ , \tag{1.13}$$

and let us denote by ℓ^w_{jk} its Weyl quantization.

Theorem 1.2 *Let $T > 0$. Let P be defined by (1.6) with real coefficients satisfying (1.9), (1.10),(1.11) and*

$$\begin{cases} (i)\ g_{jk} = \delta_{jk} + b_{jk},\ b_{jk} \in S(\langle x\rangle^{-\sigma_0}, g),\ for\ some\ \sigma_0 > 0, \\ (ii)\ a_j \in S_T(\langle x\rangle^{\frac{m}{2}}, g),\ V \in S_T(\langle x\rangle^m, g). \end{cases} \tag{1.14}$$

Then for any $\nu > 0$ one can find $C = C(\nu, T)$ such that for any $u \in C^1([0,T], \mathcal{S}(\mathbb{R}^n))$ and $f(t) = (D_t + P)u(t)$ we have

$$\sum_{j,k=1}^n \int_0^T \|\langle x\rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell^w_{jk} u(t)\|^2_{L^2} dt \leqq \ C\ (\|u(0)\|^2_{L^2} + \int_0^T \|\langle x\rangle^{\frac{1+\nu}{2}} E_{-\frac{1}{m}} f(t)\|^2_{L^2} dt).$$

Here are some remarks and examples.

Remark 1.3 *1)We know that one can find $\psi \in C_0^\infty(|x| < 1)$ and $\phi \in C_0^\infty(\frac{1}{2} \leqq |x| \leqq 2)$ positive such that $\psi(x) + \sum_{j=0}^{+\infty} \phi(2^{-j}x) = 1$, for all x in $\mathbb{R}^n$. Let $V = |x|^m \sum_{j \text{ even}} \phi(2^{-j}x) - |x|^2 \sum_{j \text{ odd}} \phi(2^{-j}x)$. Then $V \in S(\langle x\rangle^m, g)$ and since $V \geqq -|x|^2$ the operator $P = -\Delta + V$ is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$. It follows that (1.9), (1.10), (1.11) and (1.14) are satisfied, therefore Theorem 1.1 and 1.2 apply. However the lower bound (1.2) assumed in [Y-Z2] is not satisfied.*

2) Assume that $p(x,\xi) = |\xi|^2 + \varepsilon \sum_{j,k=1}^n b_{jk}(x)\xi_j\xi_k$ with $b_{jk} \in S(\langle x\rangle^{-\sigma_0}, g)$ for some $\sigma_0 > 0$. Then if ε is small enough the non trapping condition (1.11) is satisfied.

2 Proofs of the results

Let us consider the symbol $a_0(x,\xi) = \dfrac{x\cdot\xi}{\langle\xi\rangle}$. A straightforward computation shows that under condition (1.8) (ii) one can find C_0, C_1, R positive such that

$$H_p a_0(x,\xi) \geqq C_0|\xi| - C_1, \ \textit{if} \ (x,\xi)\in T^*(\mathbb{R}^n) \ \textit{and} \ |x|\geqq R. \tag{2.1}$$

where H_p denotes the Hamiltonian field of the symbol p.
Then we have the following result due to Doï [D3].

Lemma 2.1 *Assume moreover that (1.11) is satisfied then there exist* $a\in S(\langle x\rangle, g)$ *and positive constants* C_2, C_3 *such that*
(i) $H_p a(x,\xi) \geqq C_2|\xi| - C_3, \quad \forall (x,\xi)\in T^*(\mathbb{R}^n)$,
(ii) $a(x,\xi) = a_0(x,\xi), \quad$ *if* $|x|$ *is large enough.*

The symbol a is called a global escape function for p. Here is the form of this symbol. Let $\chi\in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(x) = 1$ if $|x|\leqq 1$, $\chi(x) = 0$ if $|x|\geqq 2$ and $0\leqq\chi\leqq 1$. With R large enough and $M\geqq 2R$ we have,

$$a(x,\xi) = a_0(x,\xi) + M^{\frac{1}{2}}\chi\big(\frac{x}{M}\big)a_1\big(x, \frac{\xi}{\sqrt{p(x,\xi)}}\big)(1-\theta(\sqrt{p(x,\xi)})$$

where

$$a_1(x,\xi) = -\int_0^{+\infty}\chi\big(\frac{1}{R}\pi(\Phi_t(x,\xi)\big)\, dt$$

and $\pi(\Phi_t(x,\xi)) = x(t;x,\xi)$, $\theta(t) = 1$ if $0\leqq t\leqq 1$, $\theta(t) = 0$ if $t\geqq 2$, $0\leqq\theta\leqq 1$. Details can be found in [D3].
Proof of Theorem 1.1
Let $\psi\in C^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi\in [\varepsilon, +\infty[$, $\psi(t) = 1$ in $[2\varepsilon, +\infty[$ (where $\varepsilon > 0$ is a small constant chosen later on) and $\psi'(t)\geqq 0$ for $t\in\mathbb{R}$. Following Doï [D3] we set,

$$\begin{cases} \psi_0(t) = 1-\psi(t)-\psi(-t) = 1-\psi(|t|) \\ \psi_1(t) = \psi(-t)-\psi(t) = -\operatorname{sgn} t\ \psi(|t|) \end{cases} \tag{2.2}$$

Then $\psi_j\in C^\infty(\mathbb{R})$, for $j = 0, 1$ and we have

$$\psi_0'(t) = -\operatorname{sgn} t\ \psi'(|t|) \quad \text{and} \quad \psi_1'(t) = -\psi'(|t|). \tag{2.3}$$

Let $\chi\in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ if $t\leqq\frac{1}{2}$, $\chi(t) = 0$ if $t\geqq 1$ and $\chi(t)\in[0,1]$. With a given by Lemma 2.1 we set

$$\begin{cases} \theta(x,\xi) = \dfrac{a(x,\xi)}{\langle x\rangle}, \quad (x,\xi)\in T^*(\mathbb{R}^n), \\ r(x,\xi) = \dfrac{\langle x\rangle^{\frac{m}{2}}}{\sqrt{p(x,\xi)}}, \quad (x,\xi)\in T^*(\mathbb{R}^n)\setminus 0. \end{cases} \tag{2.4}$$

Finally we set

$$-\lambda = \Big(\frac{a}{\langle x\rangle}\psi_0(\theta) - \big(M_0 - \langle a\rangle^{-\nu}\big)\psi_1(\theta)\Big)p^{\frac{1}{m}-\frac{1}{2}}\chi(r), \tag{2.5}$$

where $\nu > 0$ is an arbitrary small constant and M_0 a large constant to be chosen.
The main step of the proof is the following Lemma.

Lemma 2.2 *(i) One can find $M_0 > 0$ such that for any $\nu > 0$ there exist positive constants C, C' such that*

$$-H_p\lambda(x,\xi) \geqq C\langle x\rangle^{-1-\nu}(|\xi|^2 + |x|^m)^{\frac{1}{m}} - C', \quad \forall (x,\xi) \in T^*(\mathbb{R}^n), \tag{2.6}$$

(ii) $\lambda \in S(1, g)$,
(iii) $[P, \lambda^w] - \frac{1}{i}(H_p\lambda)^w \in Op^w S_T(1, g)$.

Proof
First of all on the support of $\chi(r)$ we have $\langle x\rangle^{\frac{m}{2}} \leqq \sqrt{p(x,\xi)} \leqq C|\xi|$. It follows that $|\xi| \sim \langle\xi\rangle$ and $|\xi| \leqq |\xi| + \langle x\rangle^{\frac{m}{2}} \leqq C'|\xi|$. Now

$$-H_p\lambda = \sum_{j=1}^{6} A_j \tag{2.7}$$

where the A_j's are defined below.
1) $A_1 = \big(H_p\langle x\rangle^{-1}\big)p^{\frac{1}{m}-\frac{1}{2}}a\psi_0(\theta)\chi(r)$. Since on the support of $\psi_0(\theta)$ we have $|a| \leqq 2\varepsilon\langle x\rangle$, it is easy to see that

$$|A_1| \leqq C_1\varepsilon\langle x\rangle^{-1}|\xi|^{\frac{2}{m}}\big(1 - \psi(|\theta|)\big)\chi(r). \tag{2.8}$$

2) $A_2 = \langle x\rangle^{-1}p^{\frac{1}{m}-\frac{1}{2}}(H_pa)\psi_0(\theta)\chi(r)$. By Lemma 2.1 (i) we have

$$A_2 \geqq C_2\langle x\rangle^{-1}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}}\big(1 - \psi(|\theta|)\big)\chi(r) - C_2'. \tag{2.9}$$

3) $A_3 = \langle x\rangle^{-1}p^{\frac{1}{m}-\frac{1}{2}}a\psi_0'(\theta)(H_p\theta)\chi(r)$. It follows from (2.3), (2.4) that

$$A_3 = -p^{\frac{1}{m}-\frac{1}{2}}|\theta|(H_p\theta)\psi'(|\theta|)\chi(r) \tag{2.10}$$

4) $A_4 = p^{\frac{1}{m}-\frac{1}{2}}(H_p\langle a\rangle^{-\nu})\psi_1(\theta)\chi(r)$. Here we have $H_p\langle a\rangle^{-\nu} = -\nu\langle a\rangle^{-2-\nu}aH_pa$. It follows from (2.2) that $A_4 = \nu p^{\frac{1}{m}-\frac{1}{2}}|a|\langle a\rangle^{-2-\nu}(H_pa)\psi(|\theta|)\chi(r)$. Now on the support of $\psi(|\theta|)$ we have $\varepsilon\langle x\rangle \leq |a|$ and since $a \in S(\langle x\rangle, g)$ we have $|a| \leq C\langle x\rangle$. It follows from Lemma 2.1 (i) that

$$A_4 \geq C_3\langle x\rangle^{-1-\nu}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}}\psi(|\theta|)\chi(r) - C_3'. \tag{2.11}$$

5) $A_5 = -p^{\frac{1}{m}-\frac{1}{2}}(M_0 - \langle a\rangle^{-\nu})(H_p\theta)\psi_1'(\theta)\chi(r)$. It follows from (2.3) that

$$A_5 = p^{\frac{1}{m}-\frac{1}{2}}(M_0 - \langle a\rangle^{-\nu})(H_p\theta)\psi'(|\theta|)\chi(r) \tag{2.12}$$

We deduce from (2.10) and (2.12) that

$$A_3 + A_5 = p^{\frac{1}{m}-\frac{1}{2}}(M_0 - \langle a\rangle^{-\nu} - |\theta|)(H_p\theta)\psi'(|\theta|)\chi(r)$$

Now $H_p\theta = \langle x\rangle^{-1}H_pa + aH_p\langle x\rangle^{-1}$. Since $|a| \leq 2\varepsilon|\theta|$ on the support of $\psi'(|\theta|)$ we deduce that $H_p\theta \geq C_4\langle x\rangle^{-1}|\xi| - C_5 \geq -C_5$. Taking $M_0 \geq 2$ and using the facts that $\psi' \geq 0$, $\chi \geq 0$ and $\varepsilon \leq |\theta| \leq 2\varepsilon$ on the support of $\psi'(|\theta|)$ we obtain

$$A_3 + A_5 \geq -C_6 \tag{2.13}$$

6) $A_6 = (\langle x\rangle^{-1}a\psi_0(\theta) - (M_0 - \langle a\rangle^{-\nu})\psi_1(\theta))p^{\frac{1}{m}-\frac{1}{2}}H_p[\chi(r)]$. We have $H_p[\chi(r)] = \dfrac{1}{\sqrt{p}}(H_p\langle x\rangle^{\frac{m}{2}})\chi'(r)$.
On the support of $\chi'(r)$ we have $\langle x\rangle \sim |\xi|^{\frac{2}{m}}$; this implies that

$$p^{\frac{1}{m}-\frac{1}{2}}|H_p[\chi(r)]| \leq C|\xi|^{\frac{2}{m}-1}\frac{|\xi|\langle x\rangle^{\frac{m}{2}-1}}{|\xi|}|\chi'(r)| \leq C_7.$$

Therefore we obtain

$$|A_6| \leq C_8. \tag{2.14}$$

Gathering the estimates obtained in (2.8) to (2.14) we obtain

$$-H_p\lambda \geq C_9\langle x\rangle^{-1-\nu}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}}\chi(r) - C_{10}. \tag{2.15}$$

Now on the support of $1-\chi(r)$ we have $|\xi| \leq C_{11}\langle x\rangle^{\frac{m}{2}}$ so $\langle x\rangle^{-1-\nu}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}} \leq C_{12}$. Therefore writing $1 = 1-\chi+\chi$ and using (2.15) we obtain (2.6).

(ii) We use the symbolic calculus in the classes $S(M,g)$. We have $\langle x\rangle^{-1} \in S(\langle x\rangle^{-1}, g)$, $a \in S(\langle x\rangle, g)$, $p \in S(\langle \xi\rangle^2, g)$ so $p^{\frac{1}{m}-\frac{1}{2}} \in S(\langle\xi\rangle^{\frac{2}{m}-1}, g)$ since $p \geq C > 0$ on $\operatorname{supp}\chi(r)$. Moreover $\chi(r) \in S(1,g)$ and on $\operatorname{supp}\chi(r)$ we have $\langle x\rangle^{\frac{m}{2}} \leq C|\xi|$. It follows that $\lambda \in S(\langle\xi\rangle^{\frac{2}{m}-1}, g) \subset S(1,g)$.

(iii) By the symbolic calculus $\{\lambda, V\} \in S_T(\langle\xi\rangle^{\frac{2}{m}-1}\langle x\rangle^m\langle x\rangle^{-1}\langle\xi\rangle^{-1}, g)$. Since we have $\langle x\rangle^{\frac{m}{2}} \leq C|\xi|$ on its support we will have $\langle x\rangle^{m-1}\langle\xi\rangle^{\frac{2}{m}-2} \leq C|\xi|^{\frac{2}{m}(m-1)}\langle\xi\rangle^{\frac{2}{m}-2} \leq C'$. Therefore $\{\lambda, V\} \in S_T(1,g)$. Now if $b \in S_T(\langle x\rangle^{\frac{m}{2}}, g)$ we have $\{\lambda, b\xi_j\} \in S(\langle\xi\rangle^{\frac{2}{m}-1}\langle x\rangle^{\frac{m}{2}}|\xi|\langle x\rangle^{-1}\langle\xi\rangle^{-1}, g)$ and since $\langle x\rangle^{\frac{m}{2}} \leq C|\xi|$ we have $\langle x\rangle^{\frac{m}{2}-1}\langle\xi\rangle^{\frac{2}{m}-1} \leq C|\xi|^{\frac{2}{m}(\frac{m}{2}-1)}\langle\xi\rangle^{\frac{2}{m}-1} \leq C'$ so $\{\lambda, b\xi_j\} \in S_T(1,g)$.

Finally $[\mathrm{Op}^w(p), \lambda^w] - \frac{1}{i}(H_p\lambda)^w \in S(\langle\xi\rangle^2\langle\xi\rangle^{\frac{2}{m}-1}\langle x\rangle^{-2}\langle\xi\rangle^{-2}, g) \subset \mathrm{Op}^w S(1,g)$. ■

End of the proof of Theorem 1.1.

Since $\lambda \in S(1,g)$ we can set $M = 1 + \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\lambda(x,\xi)|$. Let us introduce $N(t) = ((M + \lambda^w)u(t), u(t))_{L^2(\mathbb{R}^n)}$. Then there exist absolute constants $C_1 > 0, C_2 > 0$ such that $C_1\|u(t)\|^2_{L^2} \leq N(u(t)) \leq C_2\|u(t)\|^2_{L^2}$. Now

$$\frac{d}{dt}N(t) = ((M+\lambda^w)\frac{\partial u}{\partial t}(t), u(t))_{L^2} + ((M+\lambda^w)u(t), \frac{\partial u}{\partial t}(t))_{L^2}$$

Since $\dfrac{\partial u}{\partial t}(t) = -iPu(t) + if(t)$ and $P^* = P$ we obtain

$$\begin{aligned}\frac{d}{dt}N(t) =& i([P,\lambda^w]u(t),u(t))_{L^2} - 2\,\mathrm{Im}((M+\lambda^w)f(t),u(t))_{L^2}\\ =& -((-H_p\lambda)^w u(t),u(t))_{L^2} - 2\,\mathrm{Im}((M+\lambda^w)f(t),u(t))_{L^2} + O(\|u(t)\|^2_{L^2})\end{aligned}$$

By lemma 2.2 (iii).

Now by Lemma 2.2 (i) and the sharp Gårding inequality, we obtain

$$((-H_p\lambda)^w u(t),u(t))_{L^2} \geq C\|\langle x\rangle^{-\frac{1+\nu}{2}}E_{\frac{1}{m}}u(t)\|^2_{L^2} - C'\|u(t)\|^2_{L^2} \tag{2.16}$$

On the other hand we have for any $\varepsilon > 0$

$$|((M+\lambda^w)f(t),u(t))_{L^2}| \leq \varepsilon\|\langle x\rangle^{-\frac{1+\nu}{2}}E_{\frac{1}{m}}u(t)\|^2_{L^2} + C_\varepsilon\|\langle x\rangle^{\frac{1+\nu}{2}}E_{-\frac{1}{m}}f(t)\|^2_{L^2} \tag{2.17}$$

Using (2.16) and (2.17) with ε small enough, we obtain

$$\frac{d}{dt}N(t) \leq -C_1\|\langle x\rangle^{-\frac{1+\nu}{2}}E_{\frac{1}{m}}u(t)\|^2_{L^2} + C_2\|\langle x\rangle^{\frac{1+\nu}{2}}E_{-\frac{1}{m}}f(t)\|^2_{L^2} + C_3N(t)$$

Integrating this inequality between 0 and t (in [0,T]) and using Gronwall's inequality, we obtain the conclusion of Theorem 1.1. ■

Proof of theorem 1.2.

Let $\chi \in C_0^\infty(\mathbb{R}^+)$, $\chi(t) = 1$ if $t \in [0,1]$, $\chi(t) = 0$ if $t \geq 2$. Recall that according to (1.14) we have $p = |\xi|^2 + q(x,\xi)$ where $q(x,\xi) = \sum_{j,k=1}^{n} b^{jk}(x)\xi_j\xi_k$ and $b^{jk} \in S(\langle x\rangle^{-\sigma_0}, g)$. Let us set

$$A_{jk} = \frac{x_j\xi_k - x_k\xi_j}{\langle\xi\rangle},\ 1 \leq j,k \leq n \tag{2.18}$$

Then we have the following result.

Lemma 2.3 *Let a be defined in Lemma 2.1. One can find positive constants C_0, C_1 and C_2 such that if we set*

$$-\lambda = \frac{a}{(1+a^2+\sum_{j,k=1}^{n} A_{jk}^2)^{\frac{1}{2}}}p^{\frac{1}{m}-\frac{1}{2}}\chi\left(\frac{\langle x\rangle^{\frac{m}{2}}}{\sqrt{p(x,\xi)}}\right) \tag{2.19}$$

then

(i) $-H_p\lambda \geq C_0\langle x\rangle^{-3}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}}\sum_{j,k=1}^{n} A_{jk}^2 - C_1\langle x\rangle^{-1-\sigma_0}(|\xi| + \langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}} - C_2,$

(ii) $\lambda \in S(\langle\xi\rangle^{\frac{2}{m}-1}, g)$,

(iii) $[P,\lambda^w] - \dfrac{1}{i}(H_p\lambda)^w \in \mathrm{Op}^w S_T(1,g)$.

Proof

First of all we have

$$|H_pA_{jk}(x,\xi)| \leq C_1\frac{|\xi|}{\langle x\rangle^{\sigma_0}},\ 1\leq j,k\leq n,\ (x,\xi)\in T^*(\mathbb{R}^n). \tag{2.20}$$

Indeed we have $\{|\xi|^2, A_{jk}\} = 0$ and $|\{q, A_{jk}\}| \leq C_2\dfrac{|\xi|}{\langle x\rangle^{\sigma_0}}$.

Let us set

$$D = 1 + a^2 + \sum_{j,k=1}^{n} A_{jk}^2. \tag{2.21}$$

We claim that on the support of $\chi(\langle x\rangle^{\frac{m}{2}}p^{-\frac{1}{2}})$ we have

$$C_3\langle x\rangle^2 \leq D \leq C_4\langle x\rangle^2 \tag{2.22}$$

for some positive constants C_3 and C_4.

Indeed a straightforward computation shows that

$$(x.\xi)^2 + \sum_{j,k=1}^{n}(x_j\xi_k - x_k\xi_j)^2 \geq |x|^2|\xi|^2.$$

Since by Lemma 2.1 we have $a(x,\xi) = \dfrac{x.\xi}{\langle\xi\rangle}$ for $|x| \geq R_0 \gg 1$ and $|\xi| \geq C_5 > 0$ on the support of χ we deduce that $D \geq C_6\langle x\rangle^2$ when $|x| \geq R_0$. When $|x| \leq R_0$ we have $D \geq 1 \geq \dfrac{1}{1+R_0^2}\langle x\rangle^2$.

Now we can write with $r(x,\xi) = \langle x\rangle^{\frac{m}{2}}p^{-\frac{1}{2}}$,

$$\left\{\begin{array}{l} -H_p\lambda = I_1 + I_2 \\ I_1 = D^{-\frac{3}{2}}(D(H_pa) - \frac{1}{2}a(H_pD))p^{\frac{1}{m}-\frac{1}{2}}\chi(r) \\ I_2 = p^{\frac{1}{m}-\frac{1}{2}}\ aD^{-\frac{1}{2}}H_p(\chi(r)) \end{array}\right. \tag{2.23}$$

We have

$$\begin{aligned} DH_pa - \frac{1}{2}a(H_pD) =&(1 + \sum_{j,k=1}^{n} A_{jk}^2)H_pa + a^2H_pa - \frac{1}{2}a(2aH_pa + 2\sum_{j,k=1}^{n} A_{jk}H_pA_{jk}) \\ =&(1 + \sum_{j,k=1}^{n} A_{jk}^2)H_pa - a\sum_{j,k=1}^{n} A_{jk}H_pA_{jk}. \end{aligned}$$

Using (2.18) and (2.20) we see that,

$$|a| \sum_{j,k=1}^{n} |A_{jk}||H_p A_{jk}| \leq C_7 |x|^2 \frac{|\xi|}{\langle x\rangle^{\sigma_0}}. \tag{2.24}$$

Morever by Lemma 2.1 we have on the support of $\chi(r)$,

$$p^{\frac{1}{m}-\frac{1}{2}}(1+\sum_{j,k=1}^{n} A_{jk}^2)H_p a \geq (1+\sum_{j,k=1}^{n} A_{jk}^2)(C_8(|\xi|+\langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}} - C_9). \tag{2.25}$$

Therefore (2.21), (2.23), (2.24), (2.25) show that,

$$I_1 \geqq \big[C_{10}\langle x\rangle^{-3}(|\xi|+\langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}} \sum_{j,k=1}^{n} A_{jk}^2 - C_{11}\frac{|\xi|}{\langle x\rangle^{1+\sigma_0}}\big]\chi(r).$$

On the support of $1-\chi(r)$ we have $|\xi| \leq \langle x\rangle^{\frac{m}{2}}$ so we obtain,

$$I_1 \geq C_{12}\langle x\rangle^{-3}(|\xi|+\langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}} \sum_{j,k=1}^{n} A_{jk}^2 - C_{13}\frac{(|\xi|+\langle x\rangle^{\frac{m}{2}})^{\frac{2}{m}}}{\langle x\rangle^{1+\sigma_0}} - C_{14}. \tag{2.26}$$

On the other hand we have,

$$|H_p(\chi(r)| = |p^{-\frac{1}{2}}\chi'(r)H_p\langle x\rangle^{\frac{m}{2}}| \leq \frac{C_{15}}{|\xi|}|\chi'(r)||\xi|\langle x\rangle^{\frac{m}{2}-1}.$$

It follows from (2.22) and the estimate $|a| \leq C_{16}\langle x\rangle$ that,

$$|I_2| \leq C_{17}, \tag{2.27}$$

since $\langle x\rangle^{\frac{m}{2}-1}|\xi|^{\frac{2}{m}-1} \leq C_{18}$.

Then (i) in lemma 2.3 follows from (2.23), (2.26) and (2.27). The proofs of (ii) and (iii) are the same as those in the proof of lemma 2.2. ■

End of the proof of Theorem 1.2.

We introduce as before, for t in $(0,T)$.

$$N(t) = ((M_0+\lambda^w)u(t), u(t))_{L^2}$$

Where M_0 is a large constant. Then $N(t) \sim \|u(t)\|_{L^2}^2$.

Now using the equation and Lemma 2.3 (iii) we can write,

$$\frac{d}{dt}N(t) = -((-H_p\lambda)^w u(t), u(t))_{L^2} - 2\,\mathrm{Im}((M_0+\lambda^w)f(t), u(t))_{L^2} + O(\|u(t)\|_{L^2}^2)$$

Since by (1.13) and (2.19) we have $\langle x\rangle^{-2}A_{jk}^2 = \ell_{jk}^2$, Lemma 2.3 (i) and the sharp Gårding inequality ensure that

$$\begin{aligned}\frac{d}{dt}N(t) \leq & - C_1 \sum_{j,k=1}^{n} \|\langle x\rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell_{jk}^w u(t)\|_{L^2}^2 + C_2 \|\langle x\rangle^{-\frac{1+\sigma_0}{2}} E_{\frac{1}{m}} u(t)\|_{L^2}^2 \\ & + \|\langle x\rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 + C_3 N(t).\end{aligned}$$

It follows that for $0 < t < T$,

(2.28)

$$\begin{aligned}N(t) + C_1 \int_0^t \sum_{j,k=1}^{n} \|\langle x\rangle^{-\frac{1}{2}} E_{\frac{1}{m}} \ell_{jk}^w u(s)\|_{L^2}^2 ds \leq & N(0) + C_2 \int_0^T \|\langle x\rangle^{-\frac{1+\sigma_0}{2}} E_{\frac{1}{m}} u(s)\|_{L^2}^2 ds \\ & + \int_0^T \|\langle x\rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(s)\|_{L^2}^2 ds + C_3 \int_0^t N(s) ds.\end{aligned}$$

Using Theorem 1.1 to bound the second term in the right hand side and then using the Gronwall inequality we obtain

$$N(t) \leq C(T)(\|u(0)\|_{L^2}^2 + \int_0^T \|\langle x\rangle^{\frac{1+\sigma_0}{2}} E_{-\frac{1}{m}} f(t)\|_{L^2}^2 dt).$$

Using again the inequality (2.28) we obtain the conclusion of Theorem 1.2. The proof is complete. ■

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L.Robbiano : Department of Mathematics, University of Versailles, 45 Av. des Etats Unis 78035 Versailles France. email : luc.robbiano@math.uvsq.fr
C.Zuily : Department of Mathematics, University of Paris 11 91405 Orsay cedex France. email : claude.zuily@math.u-psud.fr